NORTH-HOLLAND

# Sign Pattern Matrices That Allow Orthogonality

Charles Waters
*Department of Mathematics, Astronomy and Statistics*
*Mankato State University*
*Mankato, Minnesota 56002*

Submitted by Charles R. Johnson

ABSTRACT

A pair of sign pattern row vectors (or column vectors) allows orthogonality if the two vectors are the sign patterns for two real orthogonal row vectors (respectively, column vectors). A square sign pattern matrix that does not have a zero row or zero column is sign potentially orthogonal (SPO) if every pair of rows and every pair of columns allows orthogonality. It has been conjectured that every SPO matrix $A$ allows orthogonality; that is, there is a real orthogonal matrix $H$ whose sign pattern equals $A$. A counterexample to this conjecture is presented, and sufficient conditions are given for a $\pm$ SPO matrix to allow orthogonality.

## 1. INTRODUCTION

A partial answer is given to the question of which sign pattern matrices allow orthogonality. This question was first raised by M. Fiedler, in June 1963, at the Symposium on the Theory of Graphs and Its Applications, Smolenice, Czechoslovakia. It appeared later as Problem 12 in [3]. This problem was also posed in 1991 by C. Johnson at the Conference on Qualitative Matrix Theory, Georgia State University. At this conference the question was raised as to whether every sign potentially orthogonal (SPO) sign pattern matrix allows orthogonality.

One can readily verify that all SPO matrices of dimension $n < 5$ allow orthogonality. In [2] it was proved that all $\pm$ SPO matrices of order $n < 6$

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allow orthogonality. L. Beasley and C. Johnson have each showed that all $\pm$ SPO matrices of order $n > 1$ contain a submatrix of order $n - 1$ that is SPO.

However, not every SPO matrix of order $n$ that allows orthogonality contains a SPO submatrix of order $n - 1$. The sign pattern matrix

$$\begin{bmatrix} + & - & 0 & + \\ + & + & - & 0 \\ + & 0 & + & - \\ 0 & + & + & + \end{bmatrix}$$

allows orthogonality, yet it does not have a SPO submatrix of order 3.

## 2. BASIC DEFINITIONS, NOTATION, AND OBSERVATIONS

A matrix whose entries belong to $\{0, +, -\}$ is called a sign pattern matrix. A matrix whose entries belong to $\{-, +\}$ is called a $\pm$ sign pattern matrix. If $A$ is a matrix, then $A_i$ and ${}_jA$ will denote the $i$th row and $j$th column of $A$ respectively. If $A$ is a matrix, $S$ is a collection of row coordinates of $A$, and $T$ is a collection of column coordinates of $A$, then $A[S, T]$ is the submatrix of $A$ obtained by keeping the rows of $A$ whose coordinates belong to $S$ and the columns of $A$ whose coordinates belong to $T$. The row and column coordinates of $A[S, T]$ are the same as they are in $A$. The sign pattern of a real matrix $A$ will be denoted by $Q(A)$. The transpose of a matrix $A$ will be written as $A^T$. The set of real $n \times n$ matrices over $\mathbb{R}$ will be denoted by $M_n(\mathbb{R})$. The real $n \times n$ skew-symmetric matrices will be denoted by $S_n(\mathbb{R})$. If $n$ is a positive integer, $E(n, i, j)$ is the $n \times n$ matrix whose $(k, l)$ entry is 1 if $(k, l) = (i, j)$, and 0 otherwise. In settings where no ambiguity will arise, $E(i,j)$ will denoted $E(n, i, j)$. If $A$ is a matrix and the rows and columns of $A$ are partitioned by the sets $\alpha_1, \dots, \alpha_k$ and $\beta_1, \dots, \beta_l$ respectively, then $A_{ij}\langle \alpha, \beta \rangle$ will denote the $(i, j)$ entry of the partitioned matrix. In settings where no ambiguity will arise, we will denote $A_{ij}\langle \alpha, \beta \rangle$ by $A_{ij}$. The topology on $M_n(\mathbb{R})$ will be the standard topology on $M_n(\mathbb{R})$. The topology on a subset of $M_n(\mathbb{R})$ will be the subspace topology of $M_n(\mathbb{R})$.

We will assume the following elementary results.

(1.1) Every permutation matrix is orthogonal.

(1.2) The product of two orthogonal matrices is an orthogonal matrix.

(1.3) For each square matrix $A$, and for each diagonal $d$ of $A$, there exists an equivalence of $A$, by permutation matrices, that assigns $d$ to the main diagonal.

## 3. RESULTS

EXAMPLE 3.1. Consider the sign pattern matrix

$$A = \begin{bmatrix} - & + & 0 & + & - \\ + & + & - & 0 & - \\ 0 & + & + & + & + \\ + & 0 & - & + & + \\ - & - & - & + & + \end{bmatrix}.$$

Since each pair of rows and each pair of columns has a negative product and a positive product among corresponding entries, $A$ is a SPO matrix.

Assume that $A$ allows orthogonality. Let

$$B = \begin{bmatrix} -b_{11} & b_{12} & 0 & b_{14} & -b_{15} \\ b_{21} & b_{22} & -b_{23} & 0 & -b_{25} \\ 0 & b_{32} & b_{33} & b_{34} & b_{35} \\ b_{41} & 0 & -b_{43} & b_{44} & b_{45} \\ -b_{51} & -b_{52} & -b_{53} & b_{54} & b_{55} \end{bmatrix}$$

be a real orthogonal matrix such that each $b_{ij} > 0$. Clearly the sign pattern of $B$ equals $A$. Since columns 1 and 2 of $B$ are orthogonal, $b_{11}b_{12} > b_{21}b_{22}$. Since rows 1 and 2 are orthogonal, $b_{11}b_{21} > b_{12}b_{22}$. Hence $b_{11}^2 b_{12} > b_{11}b_{21}b_{22}$ and $b_{11}b_{21}b_{22} > b_{12}b_{22}^2$. Thus $b_{11} > b_{22}$. We can apply similar arguments to $B[\{2,3\},\{2,3\}]$, $B[\{3,4\},\{3,4\}]$ and $B[\{1,4\},\{1,4\}]$ to show that $b_{22} > b_{33}$, $b_{33} > b_{44}$, and $b_{44} > b_{11}$. Hence $b_{11} > b_{11}$. Thus $A$ does not allow orthogonality.

For the remainder of this section we shall assume that $k$, $n$, and $n_1, \dots, n_k$ are positive integers, $A_{11}, \dots, A_{kk}$ are real orthogonal matrices of dimensions $n_1, \dots, n_k$ respectively,

(3.1) $n_1 + \cdots + n_k = n$,
(3.2) $A = A_{11} \oplus \cdots \oplus A_{kk}$,
(3.3) $\beta_0 = 0$, and for $i \in \{1, \dots, k-1\}$, $\beta_i = \beta_{i-1} + n_i$,
(3.4) for each $i \in \{1, \dots, k\}$, $\alpha_i = \{1, \dots, n_i\} + \{\beta_{i-1}\}$,
(3.5) if $B$ is an $n \times n$ matrix, then $\mathbf{B}$ will denote the $k \times k$ partitioned matrix obtained by partitioning the rows and columns of $B$ by $\{\alpha_1, \dots, \alpha_k\}$,
(3.6) if $B$ is an $n \times n$ matrix, then the $(i, j)$ entry of $\mathbf{B}$ is $\mathbf{B}_{ij}$, and
(3.7) $f: M_n(\mathbb{R}) \to M_n(\mathbb{R})$ is defined by $f(X) = XX^T$.

The next theorem is a consequence of well-known results about differentiable functions and continuous functions.

THEOREM 3.2. *The function $f$ is continuously differentiable, and for each $X, H \in M_n(\mathbb{R})$, $Df(X)[H] = XH^T + HX^T$.*

THEOREM 3.3. *For each $n \times n$ real orthogonal matrix $B$,* $\ker[Df(B)] = \{SB : S \in S_n(\mathbb{R})\}$, *and* $\dim\ker[Df(B)] = n(n-1)/2$.

*Proof.* Assume that $H \in \ker[Df(B)]$. Then $BH^T + HB^T = 0$. Hence $BH^T = -HB^T$. So $H = (-BH^T)B$. Let $S = -BH^T$. Then

$$S^T = (-BH^T)^T = -HB^T = BH^T = -S.$$

Conversely, if $S \in S_n(\mathbb{R})$, then $Df(B)[SB] = B(SB)^T + (SB)B^T = -S + S = 0$. In addition,

$$\begin{aligned}\dim\ker[Df(B)] &= \dim\{SB : S \in S_n(\mathbb{R})\}\\ &= \dim(\{S : S \in S_n(\mathbb{R})\}\{B\})\\ &= \dim\{S : S \in S_n(\mathbb{R})\}\\ &= n(n-1)/2.\end{aligned}$$ ∎

THEOREM 3.4. *If $H \in \ker[Df(A)]$, then for all $i, j \in \{1, \ldots, k\}$ with $i < j$, $\mathbf{H}_{ij} = -A_{ii}(\mathbf{H}_{ji})^T A_{jj}$.*

*Proof.* Assume that $H \in \ker[Df(A)]$. By Theorem 3.3 we may choose $S \in S_n(\mathbb{R})$ so that $H = SA$. Then $\mathbf{H}_{ij} = (\mathbf{S}_i)(_j\mathbf{A}) = \mathbf{S}_{ij}\mathbf{A}_{jj}$. So $\mathbf{S}_{ij} = \mathbf{H}_{ij}(A_{jj})^T$. Similarly, $\mathbf{H}_{ji} = \mathbf{S}_{ji}A_{ii}$. Thus $(\mathbf{S}_{ji})^T = A_{ii}(\mathbf{H}_{ji})^T$. Since $S$ is skew-symmetric, $-\mathbf{S}_{ij} = A_{ii}(\mathbf{H}_{ji})^T$. Then $\mathbf{H}_{ij}(A_{jj})^T = \mathbf{S}_{ij} = -A_{ii}(\mathbf{H}_{ji})^T$. Hence $\mathbf{H}_{ij} = -A_{ii}(\mathbf{H}_{ji})^T A_{jj}$. ∎

DEFINITION 3.5. Let $V = \{X \in M_n(\mathbb{R}) : x_{ij} = 0 \text{ for } i \leqslant j\}$, $W = \{Y \in M_n(\mathbb{R}) : y_{ij} = 0 \text{ for } i > j\}$, $\mathbf{V} = \{X \in M_n(\mathbb{R}) : \mathbf{X}_{ij} = 0 \text{ if } i \leqslant j\}$, and $\mathbf{W} = \{Y \in M_n(\mathbb{R}) : \mathbf{Y}_{ij} = 0 \text{ if } i > j\}$.

We shall assume that each of these sets is a real subspace of $M_n(\mathbb{R})$. Clearly, $M_n(\mathbb{R}) = V + W = \mathbf{V} + \mathbf{W}$, and $V \cap W = \mathbf{V} + \mathbf{W} = \varnothing$.

DEFINITION 3.6. Let $Z \in M_n(\mathbb{R})$, $X \in V$, and $Y \in W$ be such that $Z = X + Y$. Then $\pi_1 : M_n(\mathbb{R}) \to V$ is defined by $\pi_1(Z) = X$, and $\pi_2 : M_n(\mathbb{R}) \to W$ is defined by $\pi_2(Z) = Y$.

DEFINITION 3.7. The function $g : M_n(\mathbb{R}) \to W$ is defined by $g(Z) = \pi_2[f(Z)]$.

The next theorem is a consequence of well-known results about continuously differentiable functions.

THEOREM 3.8. *The function $g : M_n(\mathbb{R}) \to W$ is continuously differentiable, and for all $Z, H \in M_n(\mathbb{R})$, $Dg(Z)[H] = \pi_2([Df(Z)][H])$.*

THEOREM 3.9. $\dim[Dg(A)(\mathbf{W})] = \dim[W]$.

*Proof.* Assume that $Y \in \mathbf{W}$ and $Dg(A)(Y) = 0$. Then $\pi_2([Df(A)][Y]) = 0$. Since $[Df(A)][Y]$ is a symmetric matrix, $[Df(A)][Y] = 0$. Choose $S \in S_n(\mathbb{R})$ so that $Y = SA$. Since $Y_{ij} = 0$ for $i > j$, we have $Y_{ij} = 0$ for $i < j$ by Theorem 3.4. Thus $\mathbf{Y}$ is block diagonal. So $\mathbf{S}$ is block diagonal. Conversely, if $T \in S_n(\mathbb{R})$ and $\mathbf{T}$ is block diagonal, then $TA \in \mathbf{W} \cap \ker[Df(A)]$. Hence,

$$\dim\{Y \in \mathbf{W} : Dg(A)[Y] = 0\} = \sum_{i=1}^{k} \frac{n_i(n_i - 1)}{2}.$$

Thus, for $i, j \in \{1, \ldots, k\}$,

$$\begin{aligned}
\dim Dg(A)[\mathbf{W}] &= \dim \mathbf{W} - \sum_{i=1}^{k} \frac{n_i(n_i - 1)}{2} \\
&= \sum_{i=1}^{k} n_i^2 + \sum_{i<j} n_i n_j - \sum_{i=1}^{k} \frac{n_i(n_i - 1)}{2} \\
&= \frac{1}{2}\left( \sum_{i=1}^{k} n_i^2 + 2\sum_{i<j} n_i n_j \right) + \frac{n}{2} \\
&= \tfrac{1}{2}n^2 + \frac{n}{2} \\
&= \frac{n(n+1)}{2} \\
&= \dim W.
\end{aligned}$$

■

DEFINITION 3.10. For $i, j \in \{1, \ldots, k\}$, let $C_1 = \bigcup_{i<j} \alpha_i \times \alpha_j$, $C_2 = \bigcup_{i \leqslant j} \alpha_i \times \alpha_j$, $C_3 = \bigcup_{i>j} \alpha_i \times \alpha_j$. The sets $Q_1 = \text{real span}\{E(i,j) : (i,j) \in C_1\}$, $Q_2 = \text{real span}\{E(i,j) : (i,j) \in C_2\}$, and $Q_3 = \text{real span}\{E(i,j) : (i,j) \in C_3\}$.

THEOREM 3.11. *If $X \in Q_1 \cap \ker[Dg(A)]$, then $X = 0$.*

*Proof.* Assume that $X \in Q_1 \cap \ker[Dg(A)]$. Since $Q_1 = \{Y \in W : \mathbf{Y}_{ii} = 0\}$ and $\ker[Dg(A)] = \ker[Df(A)]$, we have for $i < j$, by Theorem 3.4,

$$\mathbf{X}_{ij} = -A_{ii}(\mathbf{X}_{ji})^T A_{jj} = -A_{ii}(0)^T A_{jj} = 0. \qquad \blacksquare$$

Since $\mathbf{W} = Q_2$ and $\dim Dg(A)[\mathbf{W}] = n(n+1)/2$, we may choose a subset $C$ of $C_2$ that contains $C_1$, having $n(n+1)/2$ elements, such that $\dim(Dg(A)[\text{span}\{E(i,j) : (i,j) \in C\}]) = n(n+1)/2$. We shall let $L$ denote $\text{span}\{E_{ij} : (i,j) \in C\}$, and $K = \text{span}\{E(i,j) : (i,j) \in \{1, \ldots, n\}^2 \setminus C\}$.

DEFINITION 3.12. Let $Z \in M_n(\mathbb{R})$, $X \in K$, and $Y \in L$ be such that $Z = X + Y$. Then $\Pi_1 : M_n(\mathbb{R}) \to K$ is defined by $\Pi_1(Z) = X$, and $\Pi_2 : M_n(\mathbb{R}) \to L$ is defined by $\Pi_2(Z) = Y$.

THEOREM 3.13. *There exists a neighborhood $N$ of $\Pi_1(A)$ in $K$ and a continuously differentiable function $\phi : N \to L$ such that $\phi(\Pi_1(A)) = \Pi_2(A)$ and for each $X \in N$, $g(X + \phi(X)) = I$, the $n \times n$ identity matrix. Furthermore, there exists a neighborhood $M$ of $A$ in $M_n(\mathbb{R})$ such that if $Z \in M$ and $g(Z) = I$, there exists $X \in N$ such that $Z = X + \phi(X)$.*

*Proof.* Since $g$ is continuously differentiable and $\dim Dg[L] = \dim W$, the function $\phi$, with the properties listed above, exists by the implicit function theorem. $\blacksquare$

THEOREM 3.14. *Let $S$ be an $n \times n$ sign pattern matrix that is permutationally equivalent to $B$ so that $Q(A_{ii}) = \mathbf{B}_{ii}$ for all $i \in \{1, \ldots, k\}$. If there exists $Z \in K$ and $Y \in L$ so that for all $(i,j) \in \{1, \ldots, n\}^n \setminus C$, sign $(z_{ij}) = b_{ij}$, and $b_{ij} \neq 0$ for $(i,j) \in C$, and for all $i < j$, $\mathbf{Y}_{ij} = -A_{ii}(\mathbf{Z}_{ji})^T A_{jj}$ and $Q(\mathbf{Y}_{ij}) = \mathbf{B}_{ij}$, then $S$ allows orthogonality.*

*Proof.* If suffices to show that $B$ allows orthogonality. Let $\phi : N \to L$ be the function described in Theorem 3.13. Assume that $X \in N$. Since $f(X)$ is a symmetric matrix,

$$I = g(X + \phi(X)) = \pi_2(f(X + \phi(X))) = f(X + \phi(X)).$$

Thus for all $H \in K$ and $X \in N$,

$$\begin{aligned} 0 &= Df(X + \phi(X))D(X + \phi(X))[H] \\ &= Df(X + \phi(X))(H + D\phi(X))[H]. \end{aligned}$$

Hence for all $H \in K$ and $X \in N$, $H + D\phi(X)[H] \in \ker[Df(X + \phi(X))]$.

For $H \in K$, let $P = H + D\phi(\Pi_1(A))[H]$. Then for $i, j \in \{1, \dots, k\}$ and $i < j$, $\mathbf{P}_{ij} - A_{ii}(\mathbf{P}_{ji})^T A_{jj}$. Since $H = \Pi_1(P)$, and $D\phi(\Pi_1(A))[H] = \Pi_2(P)$, for $i, j \in \{1, \dots, k\}$ and $i < j$,

$$(D\phi(\Pi_1(A))[H])_{ij} = -A_{ii}(\mathbf{H}_{ji})^T A_{jj},$$

where the rows and columns of $D\phi(\Pi_1(A))[H]$ are partitioned by $\alpha_1, \dots, \alpha_k$.

Choose $\delta > 0$ so that if $t \in (0, \delta)$, then $\Pi_1(A) + t\Pi_1(Z) \in N$. Then for $i, j \in \{1, \dots, k\}$ and $i < j$,

$$\begin{aligned} 0 &= \lim_{t \to 0^+} \frac{(\phi(\Pi_1(A) + t\Pi_1(Z)))_{ij} - (\phi(\Pi_1(A)))_{ij} - (D\phi[t\Pi_1(Z)])_{ij}}{t} \\ &= \lim_{t \to 0^+} \frac{(\phi(\Pi_1(A) + t\Pi_1(Z)))_{ij} - (\Pi_2(A))_{ij} - (D\phi[t\Pi_1(Z)])_{ij}}{t} \\ &= \lim_{t \to 0^+} \frac{(\phi(\Pi_1(A) + t\Pi_1(Z)))_{ij} + A_{ii}(t\mathbf{Z}_{ji})^T A_{jj}}{t} \\ &= \lim_{t \to 0^+} \frac{(\phi(\Pi_1(A) + t\Pi_1(Z)))_{ij}}{t} + A_{ii}(\mathbf{Z}_{ji})^T A_{jj}. \end{aligned}$$

Thus, we may choose $\delta_1 \in (0, \delta)$ so that for all $t \in (0, \delta_1)$,

$$Q\big((\phi(\Pi_1(A) + t\Pi_1(Z)))_{ij}\big) = Q\big(-A(\mathbf{Z}_{ji})^T A_{jj}\big) = Q(\mathbf{Y}_{ij}).$$

Since $\phi$ is continuous, and since for each $i \in \{1,\dots,k\}$, $Z$ is zero at each zero entry of $A_{ii}$, there exists $\delta_2 \in (0, \delta_1)$ such that if $t \in (0, \delta_2)$ then

$$Q(\phi(\Pi_1(A) + t\Pi_1(Z)))_{ii} = Q(A_{ii}).$$

Thus for all $t \in (0, \delta_2)$, $\phi(\Pi_1(A) + tZ)$ is a real orthogonal matrix whose sign pattern equals $B$. ■

COROLLARY 3.15. *Let $S$ be an $n \times n$ $\pm$ sign pattern matrix that is permutationally equivalent to a matrix $B$ such that for all $i, j \in \{1,\dots,n\}$ with $i \neq j$, $B[\{i,j\},\{i,j\}]$ allows orthogonality. Then $S$ allows orthogonality.*

*Proof.* It suffices to show that $B$ allows orthogonality. Clearly, for each $i \in \{1,\dots,n\}$, $b_{ii}$ allows orthogonality. For $i \in \{1,\dots,n\}$ let $A_{ii} = 1$ if $b_{ii} = +$, and $-1$ otherwise. Let $A = A_{11} \oplus \cdots \oplus A_{nn}$. For $i, j \in \{1,\dots,n\}$ and $i < j$, and let $Y_{ij} = y_{ij} = 1$ if $b_{ij} = +$, and $-1$ otherwise. For $i, j \in \{1,\dots,n\}$ and $i > j$, let $Z_{ij} = z_{ij} = 1$ if $b_{ij} = +$, and $-1$ otherwise. Assume that $i < j$. Since $B[\{i,j\},\{i,j\}]$ allows orthogonality, $A_{ii}Z_{ji} + A_{jj}Y_{ij} = 0$. Thus $A_{jj}^2 Y_{ij} = -A_{ii}Z_{ji}A_{jj} = -A_{ii}(Z_{ji})^T A_{jj}$. So $Q(Y_{ij}) = Q(-A_{ii}(Z_{ji})^T A_{jj})$. By Theorem 3.14, $B$ allows orthogonality. ■

COROLLARY 3.16. *Let $S$ be an $n \times n$ $\pm$* SPO *matrix which is permutationally equivalent to a matrix $B$, such that for some $i \in \{2,\dots,n\}$, $B\{[2,\dots,n\},\{2,\dots,n\}]_i = \pm B[\{1\},\{2,\dots,n\}]$, or for some $j \in \{2,\dots,n\}$, ${}_jB[\{2,\dots,n\},\{2,\dots,n\}] = \pm B[\{2,\dots,n\},\{1\}]$. Then $S$ allows orthogonality.*

*Proof.* It suffices to show that $B$ allows orthogonality. We shall prove the case in which there exists $i \in \{2,\dots,n\}$ such that $B[\{1\},\{2,\dots,n\}]$ equals $\pm B[\{2,\dots,n\},\{2,\dots,n\}]_i$. The proof of the other case follows by considering the transpose. Since $B_i$ and $B_1$ allow pairwise row orthogonality, $b_{11}$ and $b_{i1}$ are of opposite sign. By multiplying $B$ by the appropriate signature matrix, we may assume that $b_{11} = -$, $b_{k1} = +$ for $k \in \{1,\dots,n\}$, and $B[\{i\},\{2,\dots,n\}] = B[\{1\},\{2,\dots,n\}]$. Let $n_1 = 1$, $n_2 = n - 1$, and $A_{11} = -1$. Let $A_{22}$ be any real orthogonal matrix whose sign pattern is $\mathbf{B}_{22}$. Let $t > 0$. Let $Z(t) \in Q_3$ be such that $z_{i1} = 1$ and all other entries of $Z_{21}$ are equal to $t$. For $k, j \in \{2, \dots, n\}$ let $E_j = [e_k]$ be the $1 \times (n-1)$ matrix such that $e_k = 1$ if $k = j$, and 0 otherwise. For $t$ sufficiently close to 0,

$$\begin{aligned} Q(-A_{11} Z_{21}^T(t) A_{22}) &= Q\left(-A_{11}\left(\sum_{k=2}^{i-1} tE_k + E_i + \sum_{k=i+1}^{n} tE_k\right) A_{22}\right) \\ &= Q\left(E_i A_{22} - t\sum_{k=2}^{i-1} A_{11} E_k A_{22} - t \sum_{k=i+1}^{n} A_{11} E_k A_{22}\right) \\ &= Q(A_{22})_i \\ &= Q(B[\{1\},\{2,\dots,n\}]). \end{aligned}$$

By Theorem 3.14, $B$ allows orthogonality. ■

We shall say that a diagonal $\{b_{i\pi(i)} : i \in \{1, \dots, n\}\}$ of an $n \times n$ sign pattern matrix $B$ can be emphasized if, for each $\delta > 0$, there exists an $n \times n$ orthogonal matrix $A(\delta)$ such that $Q(A(\delta)) = B$, and for each $i \in \{1, \dots, n\}$, $1 - |a_{i\pi(i)}| < \delta$.

COROLLARY 3.17. *Let $S$ be an $n \times n$ $\pm$ sign pattern matrix that is permutationally equivalent to matrix $B$, such that the main diagonal of $B[\{2, \dots, n\}, \{2, \dots, n\}]$ can be emphasized, and for each $i \in \{2, \dots, n\}$, $B[\{1, i\}, \{1, i\}]$ allows orthogonality. Then $S$ allows orthogonality.*

*Proof.* It suffices to show that $B$ allows orthogonality. For each $\delta > 0$, let $A_{22}(\delta)$ be an $(n-1) \times (n-1)$ real orthogonal matrix such that $Q(A_{22}(\delta)) = B[\{2, \dots, n\}, \{2, \dots, n\}]$, and for $i \in \{2, \dots, n\}$, $1 - |a_{ii}| < \delta$. Let $n_1 = 1$ and $n_2 = n - 1$. Let $Z \in Q_3$ be such that for $i \in \{2, \dots, n\}$, $z_{i1} = 1$ if $b_{i1}$ is $+$, and $-1$ otherwise. Let $A_{11} = 1$ if $b_{11} = +$, and $-1$ otherwise. Since for each $i \in \{2, \dots, n\}$ the matrix $B[\{1, i\}, \{1, i\}]$ allows orthogonality, $b_{1i} = Q(-A_{11} z_{i1} a_{ii})$. Let $E_j$ be defined as in Corollary 3.16. Then for $\delta$ sufficiently close to 0,

$$\begin{aligned} Q(-A_{11} Z_{21}^T A_{22}(\delta)) &= Q\left(-A_{11}\left[\sum_{j=2}^{n} \sum_{i=2}^{n} z_{i1} a_{ij}(\delta) E_j\right]\right) \\ &= Q\left(\sum_{i=2}^{n} -A_{11} z_{i1} a_{ii}(\delta) E_i - \sum_{i \neq j} A_{11} z_{i1}(\delta) E_j\right) \\ &= Q(B[\{1\},\{2,\dots,n\}]). \end{aligned}$$

By Theorem 3.14, $A$ allows orthogonality. ■

## 4. EXAMPLES

One can show that all $\pm$ SPO $3 \times 3$ matrices are permutationally equivalent to

$$A = \begin{bmatrix} + & - & + \\ + & + & - \\ + & + & + \end{bmatrix}.$$

Since $A[\{2,3\},\{2,3\}]$ allows orthogonality, and $A[\{1\},\{2,3\}] = -A[\{2\},\{2,3\}]$, Corollary 3.16 implies $A$ allows orthogonality.

One can show that each $4 \times 4$ $\pm$ SPO matrix is permutationally equivalent to either

$$A = \begin{bmatrix} - & + & + & + \\ + & - & + & + \\ + & + & - & + \\ + & + & + & + \end{bmatrix}, \qquad B = \begin{bmatrix} - & + & + & + \\ + & - & + & + \\ + & + & - & + \\ + & + & + & - \end{bmatrix}, \quad \text{or}$$

$$C = \begin{bmatrix} - & - & + & + \\ + & - & + & + \\ + & + & + & - \\ + & + & + & + \end{bmatrix}.$$

Since $A[\{2,3,4\},\{2,3,4\}]$ allows orthogonality, and $A[\{1\},\{2,3,4\}] = A[\{4\},\{2,3,4\}]$, Corollary 3.16 implies that $A$ allows orthogonality.

We can permute the diagonal $\{(1,1),(2,3),(3,4),(4,2)\}$ of $B$ to the main diagonal, then apply Corollary 3.15 to show that $B$ allows orthogonality. We can apply Corollary 3.15 directly to $C$.

Consider the sign pattern matrix

$$E = \begin{bmatrix} - & - & + & + & + \\ + & - & - & + & + \\ 0 & - & + & - & 0 \\ 0 & + & + & + & - \\ + & + & + & + & + \end{bmatrix}.$$

Partition $E$ as

$$\begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix},$$

where $E_{11} = E[\{1,2\},\{1,2\}]$, $E_{12} = E[\{1,2\},\{2,3,4\}]$, $E_{21} = [\{3,4,5\},\{1,2\}]$, and $E_{22} = E[\{3,4,5\},\{3,4,5\}]$. One can verify that for all $q > 0$, sufficiently close to 0,

$$A_{11} = \begin{bmatrix} -\cos\theta & -\sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \quad \text{and } A_{22} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

are orthogonal matrices such that $Q(A_{11}) = E_{11}$ and $Q(A_{22}) = E_{22}$. Also, by using Theorem 3.4 one can show that $C$ can be chosen to be the coordinates of the starred entries of

$$H = \begin{bmatrix} 0 & * & * & * & * \\ * & 0 & * & * & * \\ 0 & 0 & * & 0 & 0 \\ 0 & 0 & * & * & 0 \\ 0 & 0 & * & * & * \end{bmatrix},$$

and that

$$Q\left(-A_{11}\begin{bmatrix} 0 & 0 & 1 \\ -2 & 1 & 1 \end{bmatrix}A_{22}\right) = \begin{bmatrix} + & + & + \\ - & + & + \end{bmatrix}.$$

By Theorem 3.14, $E$ allows orthogonality.

## 5. QUESTIONS

QUESTION 5.1. For which real $n \times n$ sign pattern matrix $S$ that allows orthogonality, is the determinant function on the set of all real $n \times n$ matrices constant on $\{G \in M_n(R) : G \text{ is orthogonal and } Q(G) = S\}$?

One can show that all orthogonal matrices of order 4 or less, with the same sign pattern, have the same determinant. We will show that all $4 \times 4$ $\pm$ real orthogonal matrices with equal sign patterns have equal determinants. It suffices to show that the det function on $4 \times 4$ real matrices is constant on each of the sets

$$\{G \in M_n(R) : G \text{ is orthogonal and } Q(G) = S\},$$

where $S \in \{A, B, C\}$, and $A$, $B$, and $C$ are the sign patterns in Section 4.

Assume that the matrix

$$F = \begin{bmatrix} -f_{11} & -f_{12} & f_{13} & f_{14} \\ f_{21} & -f_{22} & f_{23} & f_{24} \\ f_{31} & f_{32} & f_{33} & -f_{34} \\ f_{41} & f_{42} & f_{43} & f_{44} \end{bmatrix}$$

is orthogonal and that each $f_{ij} > 0$. Then $f_{22}f_{42} > f_{24}f_{44}$ and $f_{34}f_{44} > f_{32}f_{42}$. So $f_{34}f_{22}f_{42} > f_{34}f_{24}f_{44}$ and $f_{24}f_{34}f_{44} > f_{24}f_{32}f_{42}$. Hence $f_{22}f_{34} > f_{24}f_{32}$. So det $F[\{2, 3\}, \{2, 4\}]$ is positive. Expanding det $F[\{2, 3, 4\}, \{2, 3, 4\}]$ about its last row, we get that det $F[\{2, 3, 4\}, \{2, 3, 4\}]$ is negative. Hence $-f_{11}$det $F[\{2, 3, 4\}, \{2, 3, 4\}]$ is positive. Thus det $F = 1$.

A similar argument can be used to show that the determinant of any orthogonal matrix whose sign pattern is $A$ is $-1$.

Assume that the matrix

$$G = \begin{bmatrix} -g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & -g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & -g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & -g_{44} \end{bmatrix}$$

is orthogonal and that each $g_{ij} > 0$. If det $G > 0$, then $-g_{11}$ det $G[\{2, 3, 4\}, \{2, 3, 4\}]$ is positive. So det $G[\{3, 4\}, \{3, 4\}]$ is positive. Since det $G$ is positive, $-g_{21}$ det $G[\{1, 3, 4\}, \{1, 3, 4\}]$ is positive. Hence det $G[\{1, 3, 4\}, \{1, 3, 4\}]$ is negative. However, if det $G([\{1, 3, 4\}, \{1, 3, 4\}])$ is expanded about its first row, one finds that det $G[\{1, 3, 4\}, \{1, 3, 4\}]$ is positive. Then $g_{21}$ det $G([\{1, 3, 4\}, \{1, 3, 4\}])$ is negative. Hence det $G$ is negative, contradicting the assumption that det $G$ is positive. Thus det $G = -1$.

One can also show that if $A_n$ is the $n \times n$ $\pm$ matrix defined by $a_{kk} = -$ if $k \in \{2, \dots, n\}$ and $+$ otherwise, then $A_n$ allows orthogonality, and for each $n \times n$ real orthogonal matrix $G$ such that $Q(G) = A_n$, det $G = (-1)^{n+1}$.

QUESTION 5.2. Does there exist an $n \times n$ $\pm$ SPO matrix $A$ that allows orthogonality, such that the hypotheses of Theorem 3.14 are not satisfied for any block diagonal of $A$ of two or more blocks that allows orthogonality? By diagonal we mean any diagonal of any partition of the rows and columns of $A$.

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